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RESEARCH ARTICLE

Periodic Solution and Stability Behavior for Nonlinear Oscillator Having a Cubic Nonlinearity Time-Delayed

Yusry O. El-Dib

Department of Mathematics, Faculty of Education, Ain Shams University, Roxy, Egypt

* Corresponding author email: yusryeldib52@hotmail.com

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ABSTRACT

The current paper investigates the dynamics of the dissipative system with a cubic nonlinear time-delayed of the type of the damping Duffing equation. A coupling between the method of the multiple scales and the homotopy perturbation has been utilized to study the complicated dynamic problem. Through this approach, a cubic nonlinear amplitude equation resulted in at the first-order of perturbation; meanwhile, a quintic equation appears at the second-order of perturbation. These equations are combined into one nonlinear quintic Landau equation. The polar form solution is used, and linearized stability configuration is applied to the nonlinear amplitude equation. Also, a second-order approximate solution is achieved. The numerical illustrations showed that the damping, delay coefficient, and time delay play dual roles in the stability behavior. In addition, the nonlinear coefficient plays a destabilizing influence.

Keywords: Homotopy perturbation method, multiple scales method, stability analysis, damping delay oscillator Duffing equation.

1 Introduction

Through the last five decades**,** the time delayed differential equations have a wide range an modeling of dynamical phenomena in several fields in science, such as; electric, pneumatic and hydraulic networks, neuroscience, optoelectronics, as well as biological or mechanical systems, long transmission lines, robotics, etc. The description of highly potential dynamical influences caused by delayed feedback or control, aging is of great interest. Mathematically, they introduce an important class of dynamical systems to be studied by advanced mathematical techniques, involving e.g. theory of bifurcation, semi-group theory or singular perturbations.

The time delay generally appears in different control systems such as aircraft, many dynamical and electrodynamic systems, chemical or the measurements. In contrast to ordinary differential equations, the time-delay systems are of infinite dimensional in nature. The time-delay is, in many cases, a source of instability. The instability issue of control systems with delay is, therefore, both of theoretical and great practical significance. Recently, much interest has been depicted to investigate the dynamics of circuits described by delayed nonlinear equations, which exhibit chaotic attractors; they have found interesting applications in secure communications as given in Ref. [1]. The study of the dynamic behavior of such circuits is rather difficult. During the past two decades, more effort has been done in the numerical as well as theoretical analysis of uncertain systems with time-delay. Different results have been obtained to provide, for example, finite-dimensional sufficient conditions for stability/instability. Away from classical linear perturbation techniques, which depend on a small parameter, the homotopy perturbation is a new technique applied to obtain solutions regardless of the restriction to small parameters.

On the other hand, delayed differential equations are utilized to describe wide physical phenomena in engineering, physics, chemistry, biology, economics, and medicine, among others. Kruthika et al. [2] investigate the local stability of a gene-regulatory network and immunotherapy for cancer modeled in a nonlinear time-delay system. Many articles have appeared as collecting theorems homotopy methods for solutions that concerned with the properties of delayed systems [3–6]. Alomari and coworkers in [5] introduced an algorithm to find approximate analytical solutions for delay differential equations by using the homotopy analysis method and, also, the modified homotopy analysis method. They used their method to obtain an approximate solution of different linear as well as nonlinear differential equations with numerical predictions that agree well with the numerical integration solutions. Olvera et al. [6] applied an enhanced multistage HPM to solve delay differential equations having constant or variable coefficients. The method is based on a sequence of subintervals that provide approximate solutions. This method fails to investigate the stability properties. El-Dib [7, 8] applies a modulating method that based on the homotopy perturbation to study the stability behavior for strongly nonlinear oscillators.

Homotopy perturbation method is a relatively new method [9-14]. Like other methods, it has theoretical and application limitations. Some nonlinear equations are imposed without a linear variable term. The solutions of these equations lead to non-oscillation solutions. Homotopy perturbation method obtains oscillation solutions through a modification of the nonlinear equations by suggested an auxiliary term [15]. The procedure is given in [16] and a relatively comprehensive survey on the concepts, theory, and applications of homotopy perturbation method are reported through References [17, 18]. This method is used in a parameter-expansion method [19, 20, and 21]. Homotopy perturbation method with two expanding parameters has been studied by He [22]. The parameterized homotopy perturbation method has been addressed by Adamu and Ogenyi [23] for a modification of the HPM. They introduce a new parameter, alpha, which can be optimal, determined when it is equal to unity; it turns to its classic version.

The above modifications of the homotopy perturbation method cannot use for studying the stability behavior for solutions of the nonlinear equations. In Ref. [15], a modulation of the homotopy perturbation is used to investigate a nonlinear Mathieu equation. In these approaches, the arbitrary constants of the primary solution of homotopy equation are suggested to be modulated with slowness time. Therefore, one needs to improve the homotopy perturbation to allow studying the stability. Herein, we suggested a modification for the homotopy perturbation included several time-scales. In [8] El-Dib suggests a modified version of the homotopy perturbation method by absorbing the multiple scales method. This modification works especially well for nonlinear oscillators. The multiple scales method is a well-known method in the perturbation theory. It is effective for the weakly nonlinear oscillators. However, the combination of the multiple scales method with the homotopy perturbation method yields an unexpected result that used for all strongly nonlinear oscillators. This analysis is named as the homotopy-multiple-scales perturbation method. Nonlinear systems subjected to a harmonic excitation have been addressed by Nayfeh [24, 25]. Mathematically, the excitations appear either as an inhomogeneous term or as time-dependent coefficients in the governing equations. The multiple scales method is one of the important methods that avoid the secular terms in the solution, especially, at the parametric resonances. It leads to uniform expansions for the solutions [24, 25]. Therefore, it is necessary to develop and improve the homotopy perturbation to recover systems that subjected to parametric excitations and producing uniform analytical approximations. Hence, we are needed to make a matching between the multiple time-scales and the homotopy perturbation.

Herein, we introduced a homotopy-multiple-scales perturbation method. Three-time scales method is used. The second-order approximate periodic solution is achieved. This method allows finding the stability properties for a damping Mathieu equation that contains the periodic delayed parameter.

1.1 The Basic Idea for the Method of Homotopy-Multiple-Scales Perturbation (HMSP)

The principles and properties of the HPM and its applicability for several kinds of differential equations are given in varies among researchers [10-18], a general nonlinear equation is considered in the form

$$L(y)+N(y)=f(t), \tag{1.1}$$

where L is an auxiliary linear operator, N is a nonlinear operator and $f(t)$ is the inhomogeneous part. The nonlinear equation (1.1) has subject to the initial condition: $y(0)=a$ and $\dot{y}(0)=b.$ The concept of the homotopy perturbation procedure is to construct the following one-parameter family of equations:

$$H(y,\rho)=L(y)-L(u_0)+\rho\left[L(u_0)+N(y)-f(t)\right]=0, \quad \rho\in[0,1], \tag{1.2}$$

where ρ the embedding parameter is called a bookkeeping parameter and u_0 is the initial guess. The embedding parameter ρ changes from zero to unity. It is obvious that when $\rho\to 0$ Eq. (2) becomes a linear differential equation $L(y)=0,$ for which an exact solution can be calculated. As $\rho\to 1,$ it becomes the original nonlinear one. So the increasing process of ρ from zero to unity is just that of Eq (1.2) to Eq (1.1). The homotopy perturbation method depends on the homotopy parameter ρ in order to expand

$$y(t,\rho)=y_0(t)+\rho y_1(t)+\rho^2 y_2(t)+\rho^3 y_3(t)+\ldots. \tag{1.3}$$

Often, one iteration method cannot work due to the complicated nonlinear equation. At this end, we need to an additional iteration method. The perturbed for the natural frequency $\omega,$ may be useful. Using the parameter ρ to obtain an extension for the natural frequency as

$$\omega=\omega_0+\rho\omega_1+\rho^2\omega_2+\ldots\,. \tag{1.4}$$

where ω_0 is known as a linear natural frequency and, ω_j are unknowns determined from solving the solvability conditions that arise by removing the secularity? This secularity produced in each order of perturbation due to the inhomogeneity in equations describing the perturbation orders. Instead of the two expansions (1.3) and (1.4) one can get one expansion method plays the two roles. This can be achieved as follows:

If the limiting case of equation (2) when $\rho\to 0$, has the form

$$L(D)y=\frac{d^2y(t)}{dt^2}+\omega^2 y(t)=0. \tag{1.5}$$

Then, according to linear differential equations theory, the general solution of (1.5) is sought in terms of two linearly independent solutions. Assuming that, these solutions are $\cos\omega t$ and, $\sin\omega t$ so that

$$y(t)=a\cos\omega t+b\sin\omega t. \tag{1.6}$$

When $\rho>0,$ one can assume that the frequency ω^2 has been replaced by a function in ρ so that the linear harmonic equation (1.5) comes in the form

$$\frac{d^2Y_0(t,\rho)}{dt^2}+\Omega^2(\rho)Y_0(t,\rho)=0, \tag{1.7}$$

where $Y_0(t,\rho),$ is an analytical function of both t and $\rho.$ consequently, the solution of the harmonic equation (1.7) becomes

$$Y_0(t,\rho)=a\cos\Omega(\rho)t+b\sin\Omega(\rho)t. \tag{1.8}$$

Assuming that the frequency $\Omega(\rho)$ has been expanded as a power series in ρ [22] such that

$$\Omega(\rho)=\omega+\rho\omega_1+\rho^2\omega_2+\ldots, \tag{1.9}$$

where $\omega_n;\ n=1,2,\ldots$ are unknowns' arbitrary parameters? Employing the expansion (1.9) with (1.8) gets

$$Y_0(t,\rho) = a\cos\left(\omega t + \omega_1 \rho t + \omega_2 \rho^2 t + \ldots\right) + b\sin\left(\omega t + \omega_1 \rho t + \omega_2 \rho^2 t + \ldots\right) \tag{1.10}$$

Applying Trigonometry rules yields

$$\begin{aligned} Y_0(t,\rho) = &\left[a\cos\left(\omega_1 \rho t + \omega_2 \rho^2 t + \ldots\right) + b\sin\left(\omega_1 \rho t + \omega_2 \rho^2 t + \ldots\right)\right]\cos\omega t \\ &+ \left[b\cos\left(\omega_1 \rho t + \omega_2 \rho^2 t + \ldots\right) - a\sin\left(\omega_1 \rho t + \omega_2 \rho^2 t + \ldots\right)\right]\sin\omega t. \end{aligned} \tag{1.11}$$

If we recall the coefficients of both $\cos\omega t$ and $\sin\omega t$ as

$$\begin{aligned} a\cos\left(\omega_1 \rho t + \omega_2 \rho^2 t + \ldots\right) + b\sin\left(\omega_1 \rho t + \omega_2 \rho^2 t + \ldots\right) = A\left(\rho t, \rho^2 t, \rho^3 t, \ldots\right), \\ b\cos\left(\omega_1 \rho t + \omega_2 \rho^2 t + \ldots\right) - a\sin\left(\omega_1 \rho t + \omega_2 \rho^2 t + \ldots\right) = B\left(\rho t, \rho^2 t, \rho^3 t, \ldots\right) \end{aligned} \tag{1.12}$$

So that (1.11) becomes

$$Y_0(t,\rho) = A\left(\rho t, \rho^2 t, \rho^3 t, \ldots\right)\cos\omega t + B\left(\rho t, \rho^2 t, \rho^3 t, \ldots\right)\sin\omega t. \tag{1.13}$$

Clearly, both the amplitudes A and B are unknown periodic functions, in the slowness-time. Expression (1.13) represents a primary solution of the homotopy equation (1.2) with arbitrary ρ. As $\rho \to 0$ into (1.13), the result coincides with the primary solution (1.6), where $A(0) \equiv a$ and $B(0) \equiv b.$ when $\rho \to 1$, the final form of the primary solution obtained as

$$y_0(t) = A(t)\cos\omega t + B(t)\sin\omega t, \tag{1.14}$$

where $Y_0(t,1) \equiv y_0(t)$. Use the definition $T_n = \rho^n t;\ n = 0,1,2,\ \ldots,$ where T_0 represents the fasten time and T_1 refer to the slow time and T_2 refer to the slower time and so on. Therefore, (1.13) becomes

$$Y_0(t,\rho) = A(T_1, T_2, T_3, \ldots)\cos\omega T_0 + B(T_1, T_2, T_3, \ldots)\sin\omega T_0. \tag{1.15}$$

This primary solution constructed from the fasten solution and unknown slowness solutions. These functions are determined such that the solution becomes uniform.

At this end, of the view, one can assume that for $1 > \rho > 0$, the function $y(t,\rho)$ has the form $Y(T_0, T_1, T_2, \ldots, \rho)$. From the point of view of the multiple scales properties [24], the first derivative and the second derivative for a function having multi-scales may be replaced by the following expansions:

$$\frac{d}{dt} = D_0 + \rho D_1 + \rho^2 D_2 + \ldots \text{ and } \frac{d^2}{dt^2} = D_0^2 + 2\rho D_0 D_1 + \rho^2 (D_1^2 + 2D_0 D_2) + \ldots, \tag{1.16}$$

where $D_n = \dfrac{\partial}{\partial T_n}$ is used. At this end, the expansion (1.3) becomes

$$y(t,\rho) = y_0(T_0, T_1, T_2, ..) + \rho y_1(T_0, T_1, T_2, ..) + \rho^2 y_2(T_0, T_1, T_2, ..) + \ldots \tag{1.17}$$

This represents one expansion with two perturbations, one in the independent variable t and the other in the dependent variable $y(t,\rho)$. This expansion has been successfully used by Nayfeh [24, 25]. Therefore, the multiple–scales-homotopy statement can be built with zero initial guesses as

$$H(y,\rho) = L\left[y(T_0, T_1, T_2, ..)\right] + \rho\left\{N\left[y(T_0, T_1, T_2, ..)\right] - f(t)\right\} = 0, \quad \rho \in [0,1]. \tag{1.18}$$

The approximate solution for the nonlinear problem is obtained as $\rho \to 1$, thus we have

$$y_{app}(t) = \lim_{\rho \to 1} y(T_0, T_1, T_2, ..) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + \ldots \tag{1.19}$$

2 Mathematical problem

In this paper, we apply HPM [9-18, 8] for the solution of the delayed Duffing equation. In order to demonstrate how this method works, let us consider the following cubic nonlinear time-delayed Duffing oscillator, defined as

$$\frac{d^2 y}{dt^2} + 2\mu \frac{dy}{dt} + \omega^2 y + Qy^3 = \eta\left[1 - y_d^2(t-\tau)\right]y_d(t-\tau), \tag{2.1}$$

where μ the coefficient of damping term, η the delayed amplitude, τ the time delayed parameter, ω the natural frequency and y_d denote the function of the delayed influence.

Define the two parts of the nonlinear equation (2.1):

$$L(y)=\frac{d^2y}{dt^2}+\omega^2 y \quad \text{and} \quad N(y)=2\mu\frac{dy}{dt}+Qy^3-\eta\left[1-y_d^2(t-\tau)\right]y_d(t-\tau). \tag{2.2}$$

Define the homotopy parameter ρ that monotonically increases from zero to unity. Construct the homotopy state in the form:

$$H(y,\rho)=L(y)+\rho N(y)=0;\ \rho\in[0,1]. \tag{2.3}$$

Uses the time-scales $T_n=\rho^n t; n=0,1,2$ so that the function of $y(t,\rho)$ transformed to $y(T_0,T_1,T_2)$. Therefore, the homotopy-multiple –scales statement for equation (2.1) can be built with zero initial guesses as

$$H(y,\rho)=L[y(T_0,T_1,T_2)]+\rho N[y(T_0,T_1,T_2)]=0,\quad \rho\in[0,1]. \tag{2.4}$$

Using Taylor expansion, one can expand the delay function $y_d\left(T_0-\tau,T_1-\rho\tau,T_2-\rho^2\tau\right)$ as

$$y_d\left(T_0-\tau,T_1-\rho\tau,T_2-\rho^2\tau\right)=\left[1-\rho\tau D_1+\rho^2\left(\tfrac{1}{2}\tau^2D_1^2-\tau D_2\right)+\ldots\right]y_d(T_0-\tau,T_1,T_2), \tag{2.5}$$

where $D_n=\dfrac{\partial}{\partial T_n}$ is used. Insert (1.16), (1.17) and (2.5) into (2.4) the homotopy-multiple –scales statement becomes

$$\begin{aligned}H(y,\rho)&=\left(D_0^2+\omega^2\right)y+\rho\left[2(D_1+2\mu)D_0y+Qy^3-\eta\left(1-y_d^2(T_0-\tau,T_1,T_2)\right)y_d(T_0-\tau,T_1,T_2)\right]\\&+\rho^2\left[\left(D_1^2+2D_0D_2+2\mu D_1\right)y+\eta\tau D_1\left(1-3y_d^2(T_0-\tau,T_1,T_2)\right)y_d(T_0-\tau,T_1,T_2)\right]+\ldots=0\end{aligned} \tag{2.6}$$

Expand the function $y(T_0,T_1,T_2,\rho)$ as a power series in ρ

$$y(T_0,T_1,T_2,\rho)=y_0(T_0,T_1,T_2)+\rho y_1(T_0,T_1,T_2)+\rho^2y_2(T_0,T_1,T_2)+\ldots, \tag{2.7}$$

where $y_n(T_0,T_1,T_2)$ unknowns determined by are expanded the homotopy function as a power series in ρ and solving the resulting power-order equations. Also, the function $y_d(T_0-\tau,T_1,T_2,\rho)$ can be expanded as

$$y_d(T_0-\tau,T_1,T_2,\rho)=y_{d0}(T_0-\tau,T_1,T_2)+\rho y_{d1}(T_0-\tau,T_1,T_2)+\rho^2y_{d2}(T_0-\tau,T_1,T_2)+\ldots. \tag{2.8}$$

3 Sequence Solutions for the Perturbed System

In this section, we deal with obtaining the uniform three-orders-solution y_0, y_1 and y_2 then combined them to find the approximate solution. To accomplish this, we substitute (2.7) and (2.8) into equation (2.6) the primary solution of equation (2.6) has the form

$$y_0(T_0,T_1,T_2)=A(T_1,T_2)e^{i\omega T_0}+\bar{A}(T_1,T_2)e^{-i\omega T_0}. \tag{3.1}$$

Consequently, we have

$$y_{d0}(T_0-\tau,T_1,T_2)=A(T_1,T_2)e^{i\omega T_0-i\omega\tau}+\bar{A}(T_1,T_2)e^{-i\omega T_0+i\omega\tau}. \tag{3.2}$$

The first and the second-order perturbation equations are listed below:

$$\left(D_0^2+\omega^2\right)y_1=-2D_0D_1y_0-2\mu D_0y_0-Qy_0^3-\eta y_{d0}^3+\eta y_{d0}, \tag{3.3}$$

$$\begin{aligned}\left(D_0^2+\omega^2\right)y_2&=-(2D_0D_1+2\mu D_0)y_1-\left(D_1^2+2D_0D_2+2\mu D_1\right)y_0-3Qy_1y_0^2\\&-3\eta y_{d0}^2y_{d1}+3\eta\tau D_1y_{d0}^3+\eta y_{d1}-\eta\tau D_1y_{d0}.\end{aligned} \tag{3.4}$$

The uniform solution requires eliminating the secular terms that contain the factor $e^{\pm i\omega T_0}$. In order to find uniform valid expansion in this case, we must remove, the source of terms that produce secular terms in (3.3). The removing of these terms implies the following solvability conditions:

$$D_1A+\mu A+i\frac{\eta}{2\omega}e^{-i\omega\tau}A-\frac{3i}{2\omega}\left(Q+\eta e^{-i\omega\tau}\right)A^2\overline{A}=0. \tag{3.5}$$

The uniform solution of equation (3.3) has the form

$$y_1(T_0,T_1,T_2)=\frac{1}{8\omega^2}\left[\left(Q+\eta e^{-3i\omega\tau}\right)A^3e^{3i\omega T_0}+\left(Q+\eta e^{3i\omega\tau}\right)\overline{A}^3e^{-3i\omega T_0}\right] \tag{3.6}$$

Consequently,

$$y_{d1}(T_0-\tau,T_1,T_2)=\frac{1}{8\omega^2}\left[\left(Q+\eta e^{-3i\omega\tau}\right)A^3e^{3i\omega T_0}e^{-3i\omega\tau}+\left(Q+\eta e^{3i\omega\tau}\right)\overline{A}^3e^{-3i\omega T_0}e^{3i\omega\tau}\right] \tag{3.7}$$

Substituting the first-order uniform solution (3.6) into the second-order problem, the solution of equation (3.4) with no secular terms becomes

$$\begin{aligned}
y_2(T_0,T_1,T_2)=&\frac{3}{24\times8\omega^4}\left(Q+\eta e^{-3i\omega\tau}\right)\left(Q+\eta e^{-5i\omega\tau}\right)A^5e^{5i\omega T_0}-\frac{21}{64\omega^4}\left(Q+\eta e^{-3i\omega\tau}\right)^2A^4\overline{A}e^{3i\omega T_0}\\
&+\frac{3}{24\times8\omega^4}\left(Q+\eta e^{3i\omega\tau}\right)\left(Q+\eta e^{5i\omega\tau}\right)\overline{A}^5e^{-5i\omega T_0}-\frac{21}{64\omega^4}\left(Q+\eta e^{3i\omega\tau}\right)^2\overline{A}^4Ae^{-3i\omega T_0}\\
&-\frac{3i}{16\omega^3}\left(Q+\eta e^{-3i\omega\tau}+2i\omega\eta\tau e^{-3i\omega\tau}\right)\left(\mu+i\frac{\eta}{2\omega}e^{-i\omega\tau}\right)A^3e^{3i\omega T_0}\\
&+\frac{3i}{16\omega^3}\left(Q+\eta e^{3i\omega\tau}-2i\omega\eta\tau e^{3i\omega\tau}\right)\left(\mu-i\frac{\eta}{2\omega}e^{i\omega\tau}\right)\overline{A}^3e^{-3i\omega T_0}\\
&-\frac{9}{16\omega^4}i\eta\omega\tau\, e^{-3i\omega\tau}\left(Q+\eta e^{-3i\omega\tau}\right)A^4\overline{A}e^{3i\omega(T_0-\tau)}+\frac{9}{16\omega^4}i\eta\omega\tau\, e^{3i\omega\tau}\left(Q+\eta e^{3i\omega\tau}\right)\overline{A}^4Ae^{-3i\omega(T_0-\tau)}.
\end{aligned} \tag{3.8}$$

Elimination of the source of secular terms from equation (3.4) yields the following solvability condition:

$$2i\omega D_2A+D_1^2A+\left(2\mu+\eta\tau e^{-i\omega\tau}\right)D_1A-9\eta\tau D_1A^2\overline{A}e^{-i\omega\tau}+\frac{3}{8\omega^2}\left(Q+\eta e^{-3i\omega\tau}\right)\left(Q+\eta e^{-i\omega\tau}\right)A^3\overline{A}^2=0. \tag{3.9}$$

By the help of the solvability condition (3.5) one can replace the terms of D_1A, D_1^2A and $D_1\left(A^2\overline{A}\right)$ by their un-derivative equivalent terms, then (3.9) becomes

$$\begin{aligned}
&D_2A+\frac{i}{2\omega}\left[\mu^2+\eta\mu\tau e^{-i\omega\tau}+\left(1+2i\omega\tau\right)\frac{\eta^2}{4\omega^2}e^{-2i\omega\tau}\right]A+\frac{9i\eta\tau}{2\omega}e^{-i\omega\tau}\left[-3\mu+\frac{i\eta}{2\omega}\left(e^{i\omega\tau}-2e^{-i\omega\tau}\right)\right]A^2\overline{A}\\
&+\frac{3}{4\omega^2}\left(Q+\eta e^{-i\omega\tau}\right)\left[-2\mu+\eta\tau e^{-i\omega\tau}+\frac{i\eta}{2\omega}\left(e^{i\omega\tau}-3e^{-i\omega\tau}\right)\right]A^2\overline{A}\\
&-\frac{3i}{16\omega^3}\begin{bmatrix}-5Q^2+Q\eta\left(+e^{-3i\omega\tau}+6e^{i\omega\tau}-17e^{-i\omega\tau}-36i\omega\tau e^{-i\omega\tau}\right)\\+\eta^2e^{-4i\omega\tau}+6\eta^2\left(6i\omega\tau+1\right)\left(1-2e^{-2i\omega\tau}\right)\end{bmatrix}A^3\overline{A}^2=0.
\end{aligned} \tag{3.10}$$

Therefore, we are in need to combine equations (3.5) and (3.10). This combination can be achieved by multiple equations (3.5) with ρ and adding to equation (3.10) multiplied with ρ^2 resulting the first two terms in the transformation of the derivative in multiple scale method setting $\rho\to1$. At this stage the amplitude functions and the expansion of the derivative $\left(\rho D_1+\rho^2D_2\right)A(T_1,T_2)$ becomes $\frac{d}{dt}A(t)$. Finally, we obtain the following equation that governed the amplitude equation:

$$
\begin{aligned}
&\frac{d}{dt}A+\left[\mu+i\frac{\eta}{2\omega}e^{-i\omega\tau}+\frac{i}{2\omega}\left(\mu^{2}+\eta\mu\tau e^{-i\omega\tau}+(1+2i\omega\tau)\frac{\eta^{2}}{4\omega^{2}}e^{-2i\omega\tau}\right)\right]A \\
&+\frac{3}{4\omega^{2}}\left[-18i\eta\mu\omega\tau e^{-i\omega\tau}-3\eta^{2}\tau\left(1-2e^{-2i\omega\tau}\right)-\left(Q+\eta e^{-i\omega\tau}\right)\left(2\mu+2i\omega-\eta\tau e^{-i\omega\tau}-\frac{i\eta}{2\omega}\left(e^{i\omega\tau}-3e^{-i\omega\tau}\right)\right)\right]A^{2}\bar{A} \\
&-\frac{3i}{16\omega^{3}}\left[-5Q^{2}+Q\eta\left(+e^{-3i\omega\tau}+6e^{i\omega\tau}-17e^{-i\omega\tau}-36i\omega\tau e^{-i\omega\tau}\right)+\eta^{2}e^{-4i\omega\tau}+6\eta^{2}(6i\omega\tau+1)\left(1-2e^{-2i\omega\tau}\right)\right]A^{3}\bar{A}^{2}=0
\end{aligned}
\tag{3.11}
$$

This is nonlinear first-order differential equation having complex coefficients of Landau form. The solution of this equation is used to discuss the stability of the problem. This equation can be satisfied by using the following transformation:

$$
A(t)=\alpha(t)e^{-i\varpi t+i\beta(t)}, \tag{3.12}
$$

where $\alpha(t)$ and $\beta(t)$ are real functions to be determined and

$$
\varpi=\frac{1}{2\omega}\left[\mu^{2}+\eta(1+\mu\tau)\cos\omega\tau+\frac{\eta^{2}}{4\omega^{2}}(\cos 2\omega\tau+2\omega\tau\sin 2\omega\tau)\right]. \tag{3.13}
$$

Substituting (3.12) into (3.11) and separating real and imaginary parts we get

$$
\begin{aligned}
&\frac{d\alpha}{dt}+\left[\left(\mu+\frac{\eta}{2\omega}(1+\mu\tau)\sin\omega\tau\right)+\frac{\eta^{2}}{8\omega^{3}}(\sin 2\omega\tau-2\omega\tau\cos 2\omega\tau)\right]\alpha \\
&\quad+\frac{3}{8\omega^{3}}\left[\begin{array}{l}-4\eta\omega^{2}(9\mu\tau+1)\sin\omega\tau-2\eta\omega(2\mu-Q\tau)\cos\omega\tau-6\eta^{2}\omega\tau \\ +14\eta^{2}\omega\tau\cos 2\omega\tau-3\eta^{2}\sin 2\omega\tau-4Q(\mu\omega+\eta\sin\omega\tau)\end{array}\right]\alpha^{3} \\
&\quad-\frac{3\eta}{16\omega^{3}}\left[\begin{array}{l}\eta(\sin 4\omega\tau-12\sin 2\omega\tau-36\omega\tau+72\omega\tau\cos 2\omega\tau) \\ +Q(\sin 3\omega\tau+36\omega\tau\cos\omega\tau-23\sin\omega\tau)\end{array}\right]\alpha^{5}=0.
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
&\frac{d\beta}{dt}+\frac{3}{8\omega^{3}}\left[\begin{array}{l}-2Q\left(2\omega^{2}+\eta\cos\omega\tau\right)+2\eta\omega(2\mu-Q\tau)\sin\omega\tau+\eta^{2} \\ -4\eta\omega^{2}(9\mu\tau+1)\cos\omega\tau-15\eta^{2}\omega\tau\sin 2\omega\tau-3\eta^{2}\cos 2\omega\tau\end{array}\right]\alpha^{2} \\
&\quad-\frac{3}{16\omega^{3}}\left[\begin{array}{l}\eta^{2}\cos 4\omega\tau+6\eta^{2}(1-2\cos 2\omega\tau-24\omega\tau\sin 2\omega\tau) \\ +Q\eta(\cos 3\omega\tau-36\omega\tau\sin\omega\tau-11\cos\omega\tau)-5Q^{2}\end{array}\right]\alpha^{4}=0.
\end{aligned}
\tag{3.15}
$$

Equation (3.14) is a nonlinear first-order differential equation with real coefficients. Suppose it has a steady-state solution α_0 given by

$$
\begin{aligned}
&\left[\begin{array}{l}\eta(\sin 4\omega\tau-12\sin 2\omega\tau-36\omega\tau+72\omega\tau\cos 2\omega\tau) \\ +Q(\sin 3\omega\tau+36\omega\tau\cos\omega\tau-23\sin\omega\tau)\end{array}\right]\alpha_0^{4} \\
&\quad+\frac{2}{\eta}\left[\begin{array}{l}4\eta\omega^{2}(9\mu\tau+1)\sin\omega\tau+2\eta\omega(2\mu-Q\tau)\cos\omega\tau+6\eta^{2}\omega\tau \\ -14\eta^{2}\omega\tau\cos 2\omega\tau+3\eta^{2}\sin 2\omega\tau+4Q(\mu\omega+\eta\sin\omega\tau)\end{array}\right]\alpha_0^{2} \\
&\quad-\frac{16\omega^{3}}{3\eta}\left[\left(\mu+\frac{\eta}{2\omega}(1+\mu\tau)\sin\omega\tau\right)+\frac{\eta^{2}}{8\omega^{3}}(\sin 2\omega\tau-2\omega\tau\cos 2\omega\tau)\right]=0.
\end{aligned}
\tag{3.16}
$$

If we perturb the amplitude function $\alpha(t)$ around the steady-state response such that

$$
\alpha(t)=\alpha_0+\alpha_1(t), \tag{3.17}
$$

where the function $\alpha_1(t)$ represents the small deviation from the steady-state response. Substituting (3.17) into (3.14), using (3.16) and linearizing it in $\alpha_1(t)$ yields

$$\frac{d\alpha_1}{dt}-\left\{\begin{array}{l}\frac{3}{4\omega^3}\left[\begin{array}{l}-4\eta\omega^2(9\mu\tau+1)\sin\omega\tau-2\eta\omega(2\mu-Q\tau)\cos\omega\tau-6\eta^2\omega\tau\\+14\eta^2\omega\tau\cos 2\omega\tau-3\eta^2\sin 2\omega\tau-4Q(\mu\omega+\eta\sin\omega\tau)\end{array}\right]\alpha_0^2\\+4\mu+\frac{2\eta}{\omega}(1+\mu\tau)\sin\omega\tau+\frac{\eta^2}{2\omega^3}(\sin 2\omega\tau-2\omega\tau\cos 2\omega\tau)\end{array}\right\}\alpha_1=0. \tag{3.18}$$

This is a linear first-order equation which can be satisfied by

$$\alpha_1(t)=\tilde{\alpha}e^{\psi t}, \tag{3.19}$$

where $|\tilde{\alpha}|<<1$ to ensure the perturbation to be small and the constant ψ is the modulation growth rate which is given by

$$\begin{aligned}\psi=&\frac{3\alpha_0^2}{4\omega^3}\left[\begin{array}{l}-4\eta\omega^2(9\mu\tau+1)\sin\omega\tau-2\eta\omega(2\mu-Q\tau)\cos\omega\tau-6\eta^2\omega\tau\\+14\eta^2\omega\tau\cos 2\omega\tau-3\eta^2\sin 2\omega\tau-4Q(\mu\omega+\eta\sin\omega\tau)\end{array}\right]\\&+4\mu+\frac{2\eta}{\omega}(1+\mu\tau)\sin\omega\tau+\frac{\eta^2}{2\omega^3}(\sin 2\omega\tau-2\omega\tau\cos 2\omega\tau).\end{aligned} \tag{3.20}$$

Clearly, the function $\alpha_1(t)$ plays a damping role whence the constant ψ having negative values.

Substituting (3.17) into equation (3.15) and linearizing it in α_1 gets

$$\begin{aligned}&\frac{d\beta}{dt}+\frac{3\alpha_0}{8\omega^3}\left[\begin{array}{l}-2Q(2\omega^2+\eta\cos\omega\tau)+2\eta\omega(2\mu-Q\tau)\sin\omega\tau+\eta^2\\-4\eta\omega^2(9\mu\tau+1)\cos\omega\tau-15\eta^2\omega\tau\sin 2\omega\tau-3\eta^2\cos 2\omega\tau\end{array}\right](\alpha_0+2\tilde{\alpha}e^{\psi t})\\&-\frac{3\alpha_0^3}{16\omega^3}\left[\begin{array}{l}\eta^2\cos 4\omega\tau+6\eta^2(1-2\cos 2\omega\tau-24\omega\tau\sin 2\omega\tau)\\+Q\eta(\cos 3\omega\tau-36\omega\tau\sin\omega\tau-11\cos\omega\tau)-5Q^2\end{array}\right](\alpha_0+4\tilde{\alpha}e^{\psi t})=0.\end{aligned} \tag{3.21}$$

Equation (3.21) can be satisfied by

$$\beta(t)=\frac{3\alpha_0\beta_0}{4\omega^3}\left(\tfrac{1}{4}\alpha_0 t+\frac{\tilde{\alpha}e^{\psi t}}{\psi}\right), \tag{3.22}$$

where the constant β_0 is given by

$$\begin{aligned}\beta_0=&\alpha_0^2\left[\eta^2\cos 4\omega\tau+6\eta^2(1-2\cos 2\omega\tau-24\omega\tau\sin 2\omega\tau)+Q\eta(\cos 3\omega\tau-36\omega\tau\sin\omega\tau-11\cos\omega\tau)-5Q^2\right]\\&+4Q(2\omega^2+\eta\cos\omega\tau)-4\eta\omega(2\mu-Q\tau)\sin\omega\tau-2\eta^2+8\eta\omega^2(9\mu\tau+1)\cos\omega\tau\\&+30\eta^2\omega\tau\sin 2\omega\tau+6\eta^2\cos 2\omega\tau.\end{aligned} \tag{3.23}$$

Insert (3.17) and (3.19) into (3.12) using (3.23) we get

$$A(t)=(\alpha_0+\tilde{\alpha}e^{\psi t})e^{i(\beta(t)-\varpi t)}. \tag{3.24}$$

To obtain the second-order complete solution we substitute from (3.1), (3.6) and (3.7) into (2.7), using (3.24) and setting, $\rho\to 1$ hence we obtain $y_{app}(t)=y_0+y_1+y_2$. Thus, we have

$$\begin{aligned}y(t)=&2(\alpha_0+\tilde{\alpha}e^{\psi t})\cos\phi(t)+\frac{(\alpha_0+\tilde{\alpha}e^{\psi t})^3}{8\omega^3}\left[\begin{array}{l}2\omega Q\cos 3\phi(t)+3Q\mu\sin 3\phi(t)+\frac{3\eta Q}{2\omega}\cos(3\phi(t)-\omega\tau)\\+2\omega\eta(1+3\mu\tau)\cos(3\phi(t)-3\omega\tau)+3\mu\eta\sin(3\phi(t)-3\omega\tau)\\+\frac{3\eta^2}{2\omega}[\cos(3\phi(t)-4\omega\tau)-2\omega\tau\sin(3\phi(t)-4\omega\tau)]\end{array}\right]\\&+\frac{(\alpha_0+\tilde{\alpha}e^{\psi t})^5}{32\omega^4}\left[\begin{array}{l}Q^2(\cos 5\phi(t)-21\cos 3\phi(t))+\eta^2\cos(5\phi(t)-8\omega\tau)+36\omega\tau Q\eta\sin(3\phi(t)-3\omega\tau)\\+Q\eta(\cos 3\omega\tau+\cos 5\omega\tau)(\cos 5\phi(t)+\sin 5\phi(t))-42Q\eta\cos(3\phi(t)-3\omega\tau)\\-21\eta^2\cos(3\phi(t)-6\omega\tau)+36\omega\tau\eta^2\sin(3\phi(t)-6\omega\tau)\end{array}\right],\end{aligned} \tag{3.25}$$

where the function $\phi(t)$ is given by

$$\phi(t)=\frac{3\alpha_0\beta_0}{4\omega^3\psi}\tilde{\alpha}e^{\psi t}+\left(\omega-\varpi+\frac{3\alpha_0^2\beta_0}{16\omega^3}\right)t. \tag{3.26}$$

4 Numerical Estimation

The complete periodic solution (3.25) has been illustrated graphically. Several numerical calculations have been presented in the following some figures. In these graphs, the function $y(t)$ is plotted versus the variation in the independent variable t. The vertical coordinate for the distribution in the function $y(t)$ and the horizontal coordinate for the variations in the time t. in these graphs, the calculations are made for the time in the interval $0<t\le 10$. The illustrations for the influence of the damping parameter μ on the distribution of the function $y(t)$ against the variable t are the subject of figures (1)-(3). In these figures three different values for μ are considered with fixed the other parameters. The $y-curve$ plotted in Fig (1) is for the system having $\mu=0.5,\ \omega=10,\ Q=1.5,\ \eta=-10, \tau=90,\ \alpha_0=0.001$. We can observe that there exists three zeros in the $y-curve$ for $0<t\le 10$. These zeros are at the points $t_1=1.31542$, $t_2=4.75508$, $t_3=8.19474$. It is observed that $t_2-t_1=t_3-t_2=3.43966$. Further, we can distinguish the presence of two most upper tops and one most lower points in this curve. The lower point is $y(3.79)=-0.928245$. the first two most upper points are at $y(0.35)=0.928245$ and $y(7.23)=0.928244$. The magnitude of time between these two upper points is $T=6.88$.

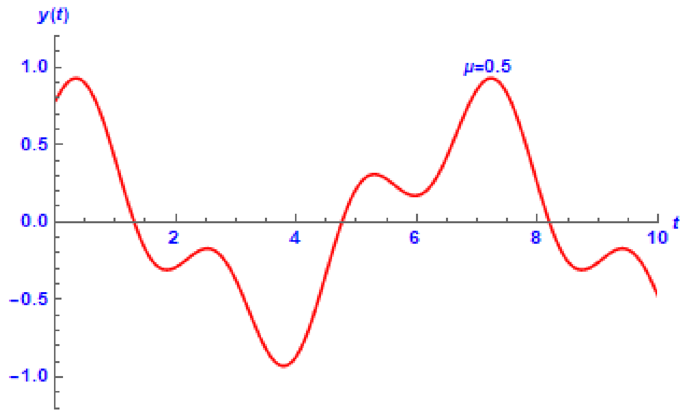

Figure 1: *The distribution of the function* $y(t)$ *against the variation in* t, *for the case of* $\mu=0.5$.

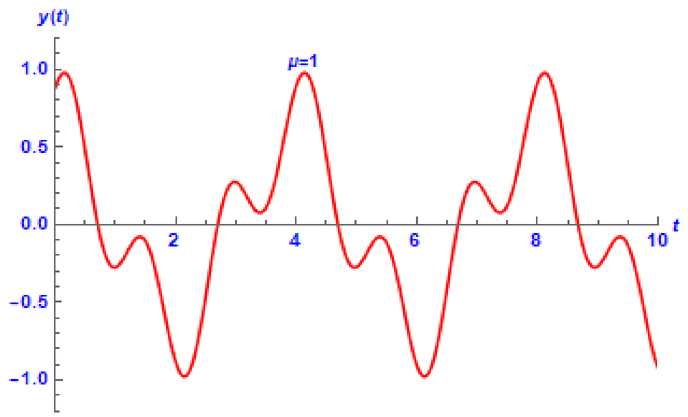

Figure 2: *The distribution of the function* $y(t)$ *against the variation in* t, *for the case of* $\mu=1$

In Fig (2) we re-plotted the same $y-curve$ in Fig (1) except that the damping parameter μ has changed to $\mu=1$. It is observed that there are five zeros points in the same $t-$interval. These zeros are at the points $t_1=0.7096$, $t_2=2.69954$, $t_3=4.68948$, $t_4=6.67942$, $t_5=8.66936$. It is found that $t_2-t_1=t_3-t_2=t_4-t_3=t_5-t_4=1.98994$. The first two most upper points are at $y_1(0.156)=0.9768245$ and $y_2(4.136)=0.976757$. The magnitude of time between these two upper points is $T=3.98$. When the damping parameter becomes $\mu=1.5$, as illustrated in Fig (3) we have the first two most upper points are at $y_1(0.088)=1.03295$ and $y_2(2.985)=1.03248$. So the time between these two upper points is $T=2.897$. The comparison between these graphs shows that the magnitude of the time T requires for occurring one period case, has decreased as μ is increased. This indicates the stabilizing influence for the parameter μ.

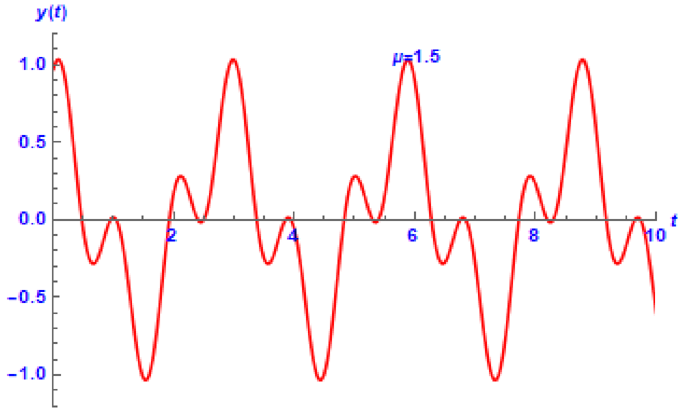

Figure 3: *The distribution of the function* $y(t)$ *against the variation in* t*, for the case of* $\mu=1.5$.

Fig (4) illustrates the change in the time delay parameter τ with fixed the other parameters. Three cases are collected in one graph for the comparison. At the case of $\tau=90$ we observe that the $y-curve$ started at the point $y(0)=0.63$. When τ becomes $\tau=100$ the starting point is $y(0)=6.31$. at the case of $\tau=110$ the starting point is $y(0)=2.05$. Two roles are observed as the time-delay is increased. A stabilizing influence as τ increased from $\tau=90$ to $\tau=100$ and destabilizing effect as τ changed from $\tau=100$ to $\tau=110$.

Fig (5) illustrates the change of the nonlinear coefficient Q on the function $y(t)$. Three cases for the nonlinear coefficient Q are presented in this graph in order to estimate the influence of increasing the nonlinear coefficient. It is observed that this increase in Q plays a destabilizing role.

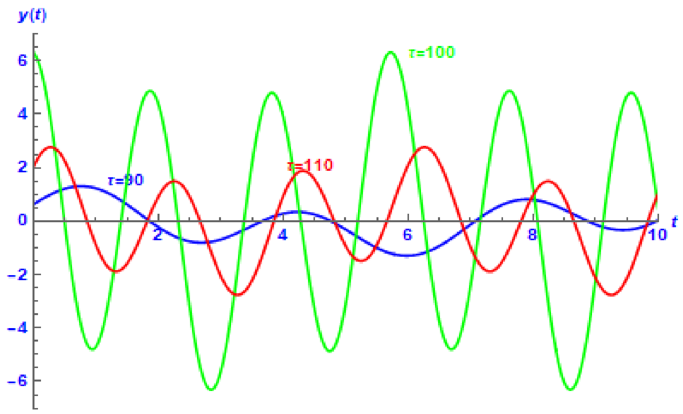

Figure 4: *The distribution of the function* $y(t)$ *versus the variation in* t*, for the cases of* $\tau=90$, 100 and 110.

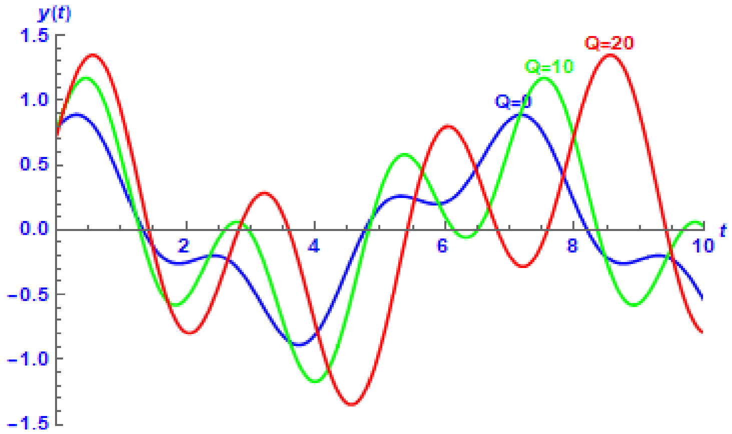

Figure 5: *The distribution of the function* $y(t)$ *versus the variation in* t, *for the cases of* $Q = 0$, 10 and 20.

5 Stability Configuration and Numerical Illustration:

The approximate solution (3.26) indicates that the stability behavior depends on the sign of the parameter ψ. The positive sign plays a growing role while the negative one plays a damping role. Eliminate α_0^2 from (3.20) by the help of (3.16) yields the following stability condition:

$$16\omega^3\eta[\eta(\sin 4\omega\tau - 12\sin 2\omega\tau - 36\omega\tau + 72\omega\tau\cos 2\omega\tau) + Q(\sin 3\omega\tau + 36\omega\tau\cos\omega\tau - 23\sin\omega\tau)]$$
$$+\frac{3}{\mu^*}\begin{bmatrix} 4\eta\omega^2(9\mu\tau+1)\sin\omega\tau + 2\eta\omega(2\mu - Q\tau)\cos\omega\tau + 6\eta^2\omega\tau \\ -14\eta^2\omega\tau\cos 2\omega\tau + 3\eta^2\sin 2\omega\tau + 4Q(\mu\omega + \eta\sin\omega\tau) \end{bmatrix}^2 < 0. \tag{4.1}$$

where the parameter μ^* is given by

$$\mu^* = \left(\mu + \frac{\eta}{2\omega}(1+\mu\tau)\sin\omega\tau\right) + \frac{\eta^2}{8\omega^3}(\sin 2\omega\tau - 2\omega\tau\cos 2\omega\tau). \tag{4.2}$$

The above stability condition (4.1) can be rearranged in terms of the Duffing coefficient Q as

$$aQ^2 + bQ + c < 0, \tag{4.3}$$

where the constants a, b and c are

$$a = \frac{12}{\mu^*}(2\mu\omega - \eta\omega\tau\cos\omega\tau + 2\eta\sin\omega\tau)^2, \tag{4.4}$$

$$b = 16\eta\omega^3(36\omega\tau\cos\omega\tau - 23\sin\omega\tau + \sin 3\omega\tau) - \frac{12}{\mu^*}\eta(\eta\omega\tau\cos\omega\tau - 2\eta\sin\omega\tau - 2\mu\omega)$$
$$\times\left[6\eta\omega\tau + 4\mu\omega\cos\omega\tau - \eta(14\omega\tau\cos 2\omega\tau - 3\sin 2\omega\tau) + 4\omega^2(1+9\mu\tau)\sin\omega\tau\right] \tag{4.5}$$

$$c = 16\eta^2\omega^3[72\omega\tau\cos 2\omega\tau - 12(3\omega\tau + \sin 2\omega\tau) + \sin 4\omega\tau]$$
$$+\frac{1}{\mu^*}\left[6\eta\omega\tau + 4\mu\omega\cos\omega\tau - \eta(14\omega\tau\cos 2\omega\tau - 3\sin 2\omega\tau) + 4\omega^2(1+9\mu\tau)\sin\omega\tau\right]^2. \tag{4.6}$$

Stability condition (4.3) can be satisfied whence

$$Q_2^* < Q < Q_1^*, \tag{4.7}$$

where $Q_1^* > Q_2^*$ and

$$Q_{1,2}^* = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}. \tag{4.8}$$

The stability condition (4.7) and the transition curves (4.8) have been illustrated numerically as shown in Figs (6), (7) and (8). The symbol "S" refers to the stable region while the symbol "U" denotes to the unstable region (resonance region). In these figures, the graph is for the $(Q-\omega)$ plane. The distributions of the Duffing coefficient Q on the vertical axis and the variation in the natural frequency ω lie on the horizontal axis. To examine the effect of a specific parameter on the stability picture, we collect three different cases for one parameter with fixed the other parameters, in each Figure.

Fig (6) deals with the distributions of the damping parameter μ with fixed of both the delayed coefficient $\eta=15$ and the parameter of delayed time $\tau=9$. The sold curve for the case of $\mu=8$, the big dash cure denotes the case of $\mu=6$ and the small dash curve refers to the case of $\mu=4$. For a specific case, the calculations show that for the zero value of the nonlinear parameter Q, the frequency ω takes the mutual roles of stability and instability. There is two stable regions S_1 and S_2. Small values of ω is stabilizing than the relatively large values. The increase of the parameter Q leads to decreasing in both S_1 and S_2 regions. Therefore, one can say that the increase in Q plays a destabilizing role. Investigation of the graph shows that the increase in the parameter μ decreasing the stable region S_1 associated with the increase in the stable region S_2. This means that the increase in the damping parameter indicates that there is a dual role in the stable configuration.

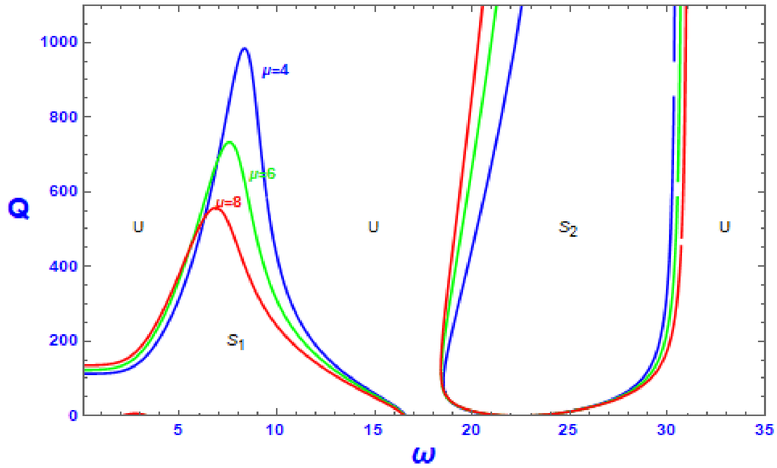

Figure 6: *The influence of the variation of the damping parameter* $(\mu=4,\ 6,\ 8)$ *for a system having* $\eta=15$, $\tau=9$.

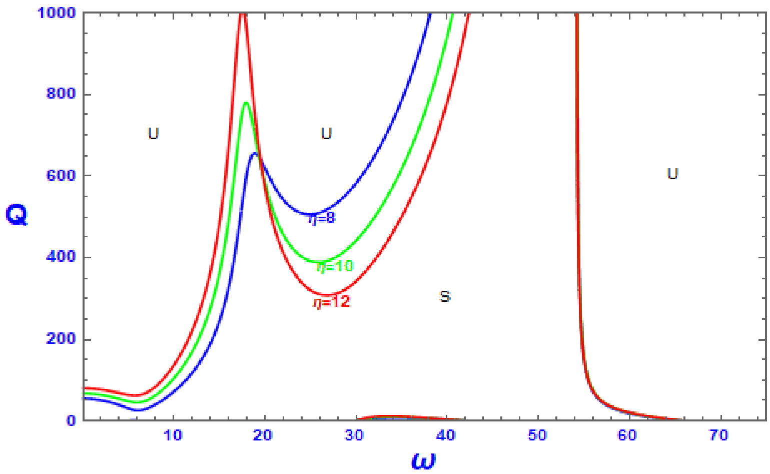

Figure 7: *The variation of the delayed coefficient,* η $(\eta=8,\ 10,\ 12)$ *with fixed and* $\tau=5$.

The examination for the influence of the coefficient of the cubic delayed term η on the stability plane is shown in Fig (7). It appears that the increase in the parameter η increases the stable region for $0<\omega<19$ and decreasing in the stable region where $19<\omega<54.$ this indicates that the distribution in η plays two roles in the stability behavior. Small stabilizing effect and relatively, large destabilizing influence. Further, notes the unstable region for ω greater than 55 has no affected with the distribution in η.

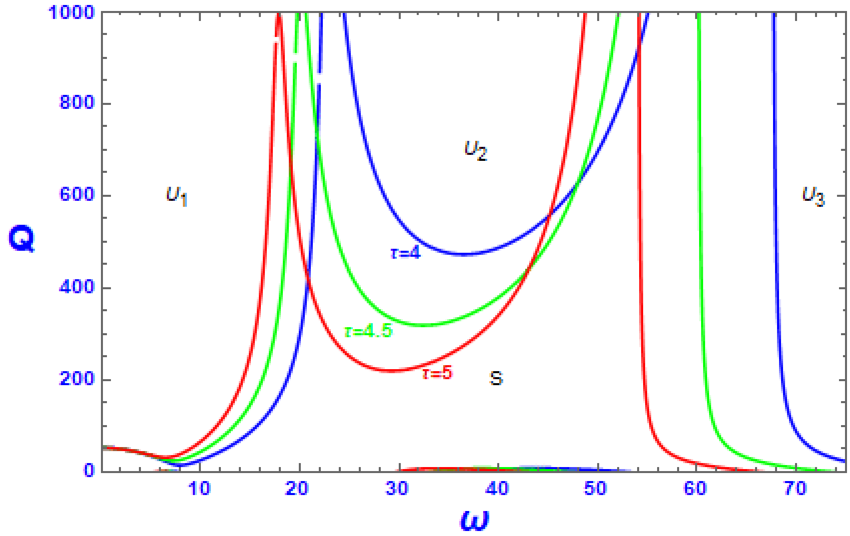

Figure 8: *The time delay distribution* $(\tau=4.0,\ 4.5,\ 5.0)$, *with fixed the parameters* $\mu=0.2$ *and* $\eta=8$.

It illustrates the behavior of the parameter τ in Fig (8). In this graph, three different cases for the delayed time τ are collected. It can distinguish three unstable regions U_1, U_2 and U_3 further, there is one stable region. It appears that the small changing in the parameter τ from $\tau=4$ to $\tau=4.5$ and then to $\tau=5$ yields more changed in both stable and unstable regions. This small variation produces a decreasing in the unstable region U_1 and increasing in both U_2 and U_3 regions. This means that the increase in τ plays a dual role in the stability examination.

6 Conclusion

In the present work, an analytical technique for the nonlinear dynamic problem with delayed self-feedback is studied. The damping Duffing oscillator with cubic nonlinear delayed term has been investigated by the use of the homotopy-multiple-scales method. This method can be used as a powerful mathematical tool for studying the stability of the solution of nonlinear oscillator systems arising in nonlinear science and engineering. Two solvability conditions obtained one of them is a cubic nonlinear first-order equation in the slow variable and the second one is a quintic nonlinear first-order in the slower variable. These nonlinear equations are combined in one equation through the concept of the homotopy method. Consequently, the amplitude equation, in the form of Landau equation, with the cubic and the quantic nonlinear terms is constructed. Polar form solution is used and enabling to discuss the linear stability around the steady state resonance. The numerical illustration showed that the parameters ω, μ, η and τ plays a dual role in the nonlinear stability.

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References

[1] B.Sheu, M. Ismail, E. Sanchez-Sinencio, and T.H. Wu, “Microsystems Technology for Multimedia Applications: An introduction”, *New Jerstey:IEEE Press*, 1995.

[2] Kruthika H.A., Mahindrakar, A. D. and Pasumarthy, R. “Stability Analysis of Nonlinear Time–Delayed Systems with Application to Biological Models”, *Int. J. Appl. Math. Comput. Sci.*, Vol. 27, No. 1, pp. 91–103, 2017.

[3] Shakeri, F. and Dehghan, M. "Solution of delay differential equations via a homotopy perturbation method", *Mathematical and Computer Modelling*, vol. 48, no. 3-4, pp. 486–498, 2008.

[4] Wu, Z.-K. "Solution of the enso delayed oscillator with homotopy analysis method," *Journal of Hydrodynamics*, vol. 21, no. 1, (),pp. 131–135,2009.

[5] Alomari, A. K. ,Noorani, M. S. and Nazar, R. "Solution of delay differential equation by means of homotopy analysis method", *Acta Applicandae Mathematicae*, vol. 108, no. 2, ,pp. 395–412, 2009.

[6] Olvera, D., Elías-Zúñiga, A., López de Lacalle, L. N. and Rodríguez, C. A.," Approximate Solutions of Delay Differential Equations with Constant and Variable Coefficients by the Enhanced Multistage Homotopy Perturbation Method", *Abstract and Applied Analysis*, Article ID 382475, pp.1-12, 2015.

[7] El-Dib, Y.O, "Homotopy Perturbation for Excited Nonlinear Equations", *Science and* Engineering Applications, Vol. 2(1), pp 96-108, 2017.

[8] El-Dib, Y.O., "Multiple scales homotopy perturbation method for nonlinear oscillators", *Nonlinear Sci. Lett. A*, Vol.8, No.4, pp. 352-364, 2017.

[9] G. Papy, "Topologieals grundlage des analysis-unterrichts Vandenholck and Ruprecht", *Göttingen*, Germany,1970.

[10] He, J.H., "Homotopy Perturbation Technique", *Computational Methods in Applied Mechanics and Engineering*, 178, pp. 257-262, 1999.

[11] He, J.H., "Homotopy Perturbation Method: a new nonlinear analytical technique", *Applied* Mathematics and Computation, vol. 135, pp. 73–79, 2003.

[12] He, J.H., "Homotopy Perturbation method and homotopy analysis method", Applied Mathematics and Computation,vol. 156, pp.527–539, 2004.

[13] He, J.H., "New interpretation of homotopy perturbation method", *International Journal of Modern Physics B*, vol. 20,pp. 2561–2568, 2006.

[14] Liao ,S.J. and Chwang,A.T, "Application of homotopy analysis method in nonlinear oscillations", *ASME J. Appl. Mech.*, 65 , pp. 914–922, 1998.

[15] He, J.H.," Homotopy Perturbation Method with an Auxiliary Term", *Abstract and Applied Analysis*, 2012, Article Number 857612.

[16] He, J H, "New interpretation of homotopy perturbation method", *Int. J. Mod. Phys. B* 20, pp. 2561-2568, 2006.

[17] He, J. H, "Some asymptotic methods for strongly nonlinear equations", *Int. J. Mod. Phys. B.* 20, pp. 1141-1199, 2006.

[18] He, J. H, "An elementary introduction to recently developed asymptotic methods and nanomechanics in textile engineering", .*Int. J. Mod. Phys. B.* 22, pp.3487-3578, 2008.

[19] Wang S.Q. and He, J H, "Nonlinear oscillator with discontinuity by parameter-expansion Method", *Chaos Soliton. Fract.* 35, pp.688-691,2008.

[20] L Xu,L "Application of He's parameter-expansion method to an oscillation of a mass attached to a stretched elastic wire", *Phys. Lett. A* 368, pp. 259-262,2007.

[21] Shou, D H and He, J H, "Application of Parameter-expanding Method to Strongly Nonlinear Oscillators" , *Int. J. Nonlin. Sci. Num.* 8, no 1, pp.121-124, 2007.

[22] He, J.H., "Homotopy perturbation method with two expanding parameters", *Indian Journal of Physics*, 88(2), pp. 193-196, 2014.

[23] Adamu, M.Y. and Ogenyi, P.," The parameterized homotopy perturbation method", *Nonlinear Sci. Lett. A*, Vol. 8(2), pp. 240-243, 2017.

[24] Nayfeh ,A. H. and Mook, D. T. "Nonlinear Oscillations", *Wiely-Interscience, New York*, 1979.

[25] Nayfeh, A. H. "Problems in Perturbation". *Second Edition, Wiley, New York*, 1993.